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ACM BUNDLES ON GENERAL HYPERSURFACES IN $\mathbb{P}^5$ OF LOW DEGREE

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Abstract. In this paper we show that on a general hypersurface of degree $r = 3, 4, 5, 6$ in $\mathbb{P}^5$ a rank 2 vector bundle $\mathcal{E}$ splits if and only if $h^1\mathcal{E}(n) = h^2\mathcal{E}(n) = 0$ for all $n \in \mathbb{Z}$. Similar results for $r = 1, 2$ were obtained in [15], [16] and [1].

1. Introduction

The construction of rank 2 bundles on smooth varieties X of dimension $n > 3$ is strictly related with the structure of subvarieties of codimension 2. When X is a projective space, then there are few examples of these subvarieties which are smooth. The famous Hartshorne's conjecture suggests that all smooth subvarieties of codimension 2 in $\mathbb{P}^7$ are complete intersection. Rephrased in the language of vector bundles, this means that all rank 2 bundles on $\mathbb{P}^7$ decompose in a sum of two line bundles.

Also in $\mathbb{P}^5$, $\mathbb{P}^6$, we do not have examples of indecomposable rank 2 bundles. In $\mathbb{P}^4$, only the Horrocks-Mumford's indecomposable bundle is known. This bundle has some non–zero cohomology group, since it is well known that a rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^r$ ($r \geq 3$) splits if and only if it is "arithmeticall Cohen–Macaulay" (ACM for short), i.e. $h^i(\mathcal{E}(n)) = 0$ for all $n, i = 1, \ldots, r-1$.

ACM property does not imply a decomposition when we replace the projective space with other smooth threefolds. There are examples of indecomposable ACM bundles of rank 2 on a general hypersurface of degree $r = 2, 3, 4, 5$ in $\mathbb{P}^4$. On the other hand we proved in [6] that all ACM rank 2 bundles on a *general* sextic in $\mathbb{P}^4$ splits.

In this paper we examine the similar problem for general hyperpsurfaces X in $\mathbb{P}^5$, in some sense the easiest examples of smooth 4-folds different from $\mathbb{P}^4$.

It is well known that a general quadric hypersurface X in $\mathbb{P}^5$ contains families of planes. Since any plane S has a canonical class which is a twist of the restriction of the canonical class of the quadric (in other words: a plane is "subcanonical" in X), then S corresponds via the Serre's construction to a rank 2 bundle $\mathcal{E}$ on X which is indecomposable (for S is not complete intersection of X and some other hypersurface) and ACM (for S is arithmetically Cohen–Macaulay).

On the other hand, since any indecomposable ACM rank 2 bundle on a general sextic hypersurface in $\mathbb{P}^5$ would restrict to an indecomposable rank 2 ACM bundle on a general hyperplane section of X, which is a general sextic hypersurface of $\mathbb{P}^4$, then by the main result of [6] we know that such bundles cannot exist (see proposition 3.6 below).

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Thus we are led to consider *general* hypersurfaces $X_r \subset \mathbb{P}^5$ of low degree r and study ACM rank 2 bundles on X_r. Our main result shows that none of such vector bundles lives on X_r for $2 < r < 7$:

Theorem 1.1. *Let $\mathcal{E}$ be a rank 2 vector bundle on a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r = 3, 4, 5, 6$. Then $\mathcal{E}$ splits if and only if*

$$h^i(\mathcal{E}(n)) = 0 \qquad \forall n \in \mathbb{Z} \qquad i = 1, 2.$$

Notice that one finds indecomposable ACM rank 2 bundles on general hypersurfaces of degree $3, 4, 5$ in $\mathbb{P}^4$. So we prove in fact that they do not lift from a general hyperplane section of X to X itself.

The proof is achieved using the tools of [6], since we have a classification of possible ACM indecomposable rank 2 bundles on a general hypersurface of low degree in $\mathbb{P}^4$. It has been obtained by Arrondo and Costa in degree 3 (see [2]), by the second author in degree 4 (see [13]) and by both authors in degree 5 ([5]). This implies a numerical characterization of the possible Chern classes of indecomposable ACM bundles of rank 2 on X (see also [12]), and we conclude with a case by case analysis.

In the language of codimension 2 subvarieties, we get the following characterization of complete intersections, which is the analogue of the classical Gherardelli's criterion for curves in $\mathbb{P}^3$:

Corollary 1.2. *Let S be a surface contained in a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r = 3, 4, 5, 6$. Then S is complete intersection in X_r if and only if it is subcanonical (i.e. its canonical class ω_X is $\mathcal{O}_S(e)$, for some $e \in \mathbb{Z}$) and $h^i\mathcal{I}_{S/X_r}(n) = 0$ for all $n \in \mathbb{Z}$ and $i = 1, 2$, where $\mathcal{I}_{S/X_r}$ is the ideal sheaf defining S in X_r.*

Let us finish with some remarks.

The non–existence of indecomposable ACM rank 2 bundles on hypersurfaces of degree $r \geq 7$ in $\mathbb{P}^4$ has not been settled yet simply because the tecnicalities introduced in [6] become odd as the degree r grows. Indeed first of all the number of Chern classes which are not excluded using the main result of [12] grows as a linear function of r. Furthermore, as r grows, for any value of c_1 one has to exclude an increasing number of second Chern classes. This is easy when c_2 is big, but becomes hard for low c_2 (compare the proof of case 5.11 in [6]), as we have to exclude the existence of some curves on X, which could be reducible or even non–reduced. We did not find a general argument for this step: only a careful ad hoc examination led us to conclude the case of sextic threefolds in $\mathbb{P}^4$.

On the other hand, there are strong evidences that ACM rank 2 indecomposable bundles cannot exist on general hypersurfaces of degree 7 or more. We were unable to prove this statement in $\mathbb{P}^4$. Could it be easier to find a direct proof for hypersurfaces in $\mathbb{P}^5$?

In any event, the main theorem implies easily:

Corollary 1.3. *On a general hypersurface X of degree $3, 4, 5, 6$ in $\mathbb{P}^n$, $n \geq 5$, a rank 2 vector bundle splits if and only if it is arithmetically Cohen–Macaulay.*

Finally observe that Evans and Griffith proved in [8] that a rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^4$ splits if and only $h^1(\mathcal{E}(n)) = 0$ for all n. This condition is considerably weaker than ACM. We wonder if a similar result could work on a general hypersurface of low degree in $\mathbb{P}^5$.

2. Notations and generalities

We work over the field of complex numbers $\mathbb{C}$. Let $X_r \subset \mathbb{P}^5$ be a smooth 4–dimensional hypersurface of degree $r \geq 1$. The letter H will denote the class of a hyperplane section of X_r. We have $Pic(X_r) \cong \mathbb{Z}[H]$, and $H^4 = r$. Recall that the canonical class of X_r is $\omega_{X_r} = (r-6)H$. Given a vector bundle $\mathcal{E}$ on X_r we introduce the non negative integer

$$b(\mathcal{E}) = b = \max\{n \mid h^0(\mathcal{E}(-n)) \neq 0\}. \tag{2.1}$$

Definition 2.1. *We say that the vector bundle $\mathcal{E}$ is normalized if $b(\mathcal{E}) = 0$.*

Notice that changing $\mathcal{E}$ with $\mathcal{E}(-b)$, we may always assume that $\mathcal{E}$ is normalized. From now on we will assume this.

We denote by $c_1 = c_1(\mathcal{E})$ the first Chern class of $\mathcal{E}$ identified with an integer via the isomorphism $Pic(X_r) \cong \mathbb{Z}[H]$.

When $\mathcal{E}$ has rank 2, the number

$$2b - c_1 = 2b(\mathcal{E}) - c_1(\mathcal{E}) \tag{2.2}$$

is invariant by twisting and measures the "level of stability of $\mathcal{E}$". Indeed $\mathcal{E}$ is stable (semistable) if an only if $0 > 2b - c_1$ ($0 \geq 2b - c_1$).

We say that $\mathcal{E}$ is "arithmetically Cohen–Macaulay (ACM)" when for all $n \in \mathbb{Z}$ we have $h^1(\mathcal{E}(n)) = h^2(\mathcal{E}(n)) = 0$. Clearly this implies, by duality, $h^3(\mathcal{E}(n)) = 0$ for all $n \in \mathbb{Z}$.

Take a global section s of $\mathcal{E}$ whose zero-locus S has codimension 2. We have the following exact sequence (see e.g. [17]):

$$0 \to \mathcal{O}_{X_r} \to \mathcal{E} \to \mathcal{I}_{S/X_r}(c_1(\mathcal{E})) \to 0 \tag{2.3}$$

which relates the cohomology of $\mathcal{E}$ with the geometric properties of $S \subset X_r$ encoded by the cohomology groups of the ideal sheaf $\mathcal{I}_{S/X_r}$ of S.

In particular S is subcanonical, i.e. $K_S \cong \mathcal{O}_S(c_1(\mathcal{E}) + r - 6)$, moreover $c_2(\mathcal{E}) = \deg S$. Also we have a formula for the sectional genus g of the surface S:

$$2g - 2 = c_2 + K_S \cdot H \cdot S = c_2 + (c_1 + r - 6)H \cdot H \cdot S = c_2(c_1 + r - 5) \tag{2.4}$$

Conversely, starting with a locally complete intersection and subcanonical surface S contained in X_r one can reconstruct a rank 2 vector bundle having a global section vanishing exactly on S. In these cases we will say that the vector bundle "$\mathcal{E}$ is associated with S".

We notice that when $\mathcal{E}$ is normalized, then every global section of $\mathcal{E}$ has zero-locus of codimension 2.

If Y_r is a general hyperplane section of X_r and $\mathcal{E}$ is a rank two vector bundle on X_r, we denote by $\mathcal{E}'$ the restriction of $\mathcal{E}$ to Y_r. We know that $Pic(Y_r) \cong Z[h]$, where h is the hyperplane class of Y_r. Under the isomorphism $Pic(X_r) \cong Pic(Y_r)$ we have $c_1(\mathcal{E}) = c_1(\mathcal{E}')$ and $c_2(\mathcal{E}) = c_2(\mathcal{E}')$.

We recall here the main results of [12] and [6], which we are going to use several times in the sequel:

Theorem 2.2. *(see [12]) Let Y_r be a smooth hypersurface of degree r in $\mathbb{P}^4$. If $\mathcal{E}$ is an ACM and normalized rank 2 vector bundle on Y_r, then $\mathcal{E}$ splits unless $r > c_1 > 2 - r$.*

Theorem 2.3. *(see [6]) Let Y be a general hypersurface of degree 6 in $\mathbb{P}^4$. Then a rank 2 vector bundle $\mathcal{E}$ on Y splits in a sum of line bundles if and only if $\mathcal{E}$ is ACM.*

3. Some preliminary general results

Remark 3.1. Consider the exact sequence which links $\mathcal{E}$ with its restriction $\mathcal{E}'$ to a general hyperplane section Y_r of X_r:

$$0 \to \mathcal{E}(-1) \to \mathcal{E} \to \mathcal{E}' \to 0 \tag{3.1}$$

Then $b(\mathcal{E}') \geq b(\mathcal{E})$ and equality holds when $h^1(\mathcal{E}(-b(\mathcal{E})-2)) = 0$, which is true when $\mathcal{E}$ is ACM.

Notice that, by the sequence, if $\mathcal{E}$ is ACM on X_r then $\mathcal{E}'$ is also ACM on Y_r. It is clear that $\mathcal{E}'$ splits when $\mathcal{E}$ splits. Conversely assume that $\mathcal{E}$ is ACM and $\mathcal{E}'$ splits. Take a global section $s' \in H^0(\mathcal{E}'(a))$ with empty zero-locus. The surjection $H^0(\mathcal{E}(a)) \to H^0(\mathcal{E}'(a)) \to 0$ derived from sequence (3.1) shows that s' lifts to a global section $s \in H^0(\mathcal{E}(a))$, whose zero-locus must be empty, since otherwise it had at most codimension 2, a contradiction for it does not intersect a hyperplane.

It follows that we may apply the main result of [12], getting:

Proposition 3.2. *If $\mathcal{E}$ is an ACM and normalized rank 2 vector bundle on X_r, then $\mathcal{E}$ splits unless $r > c_1 > 2 - r$.*

Some well known facts about the non–existence of surfaces of low degree on general hypersurfaces of $\mathbb{P}^5$ together with a numerical analysis leads us to the following refinement of the previous result:

Proposition 3.3. *Let $\mathcal{E}$ be a normalized ACM rank 2 vector bundle on a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r \geq 3$. Then $\mathcal{E}$ splits unless $3 - r < c_1 < r$.*

Proof. We need to exclude the case $c_1 = 3 - r$.
Consider a global section s and its zero-locus S. The exact sequence (2.3) here reads

$$0 \to \mathcal{O}_{X_r} \to \mathcal{E} \to \mathcal{I}_{S/X_r}(3-r) \to 0$$

and implies $h^0(\mathcal{E}(r-3)) = h^0(\mathcal{O}_{X_r}(r-3))$. By Serre duality $h^4(\mathcal{E}(r-3)) = h^0(\mathcal{E}(r-6)) = h^0(\mathcal{O}_{X_r}(r-6))$. Moreover $h^1(\mathcal{E}(r-3)) = h^2(\mathcal{E}(r-3)) = 0$ for $\mathcal{E}$ is ACM. Thus $\chi(\mathcal{E}(r-3)) = h^0(\mathcal{O}_{X_r}(r-3)) + h^0(\mathcal{O}_{X_r}(r-6))$. By Riemann-Roch one is thus able to compute the second Chern class of $\mathcal{E}(r-3)$, hence also the second Chern class c_2 of $\mathcal{E}$. It turns out $c_2 = 1$. So S is a plane. Since a general X_r of degree $r \geq 3$ contains no planes (see e.g. [7]), then X_r has no indecomposable and normalized rank 2 ACM bundles with $c_1 = -r + 3$. □

Next we use the link between ACM bundles with $c_1 = r - 1$ and pfaffian hypersurfaces.

Definition 3.4. *A hypersurface $X_r \subset \mathbb{P}^5$ is pfaffian if its equation is pfaffian of a skew-symmetric matrix of linear forms in $\mathbb{P}^5$.*

The results proved by Beauville in [3] exclude the existence of ACM rank 2 bundles with $c_1 = r - 1$ on a general hypersurface X_r, $r \geq 3$.

Proposition 3.5. *When $r \geq 3$ then a general hypersurface $X_r \subset \mathbb{P}^5$ has no normalized indecomposable rank 2 ACM bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = r - 1$ and $c_2 = r(r-1)(2r-1)/6$.*

Proof. It follows soon by the following two facts proved in [3]. X_r is pfaffian if and only if there exists an indecomposable ACM rank 2 vector bundle on X_r with Chern classes as in the statement. Moreover the general hypersurface of degree $r \geq 3$ in $\mathbb{P}^5$ is not pfaffian. □

Let us now turn to hypersurfaces of low degree. We want to exclude the existence of indecomposable ACM rank 2 bundles on general hypersurfaces. This follows easily on sextic hypersurfaces, using the main result of [6].

Proposition 3.6. *On a general sextic hypersurface $X \subset \mathbb{P}^5$ all ACM rank 2 bundles $\mathcal{E}$ split.*

Proof. A general hyperplane section Y of X is a general sextic hypersurface of $\mathbb{P}^4$. By remark 3.1 we know that an indecomposable ACM rank 2 bundle on X restricts to an indecomposable ACM rank 2 bundle on Y. In [6] we excluded the existence of such bundles. □

For hypersurfaces of degree $r < 6$ we cannot use the same procedure, since there exist indecomposable ACM rank 2 bundles on general cubics, quartics and quintics of $\mathbb{P}^4$.

We use instead an examination of the family of surfaces associated to ACM rank 2 bundles. Let us set some more pieces of notation.

Call $H(d,g)$ the Hilbert scheme of arithmetically Cohen–Macaulay (ACM) surfaces in $\mathbb{P}^5$ of degree $d = c_2$ and sectional genus g such that $2g-2 = c_2(c_1+r-5)$. This is a smooth quasi-projective subvariety of the Hilbert scheme.
Let $\mathbb{P}(r)$ be the scheme which parametrizes hypersurfaces of degree r in $\mathbb{P}^5$.
In the product $H(d,g) \times \mathbb{P}(r)$ one has the incidence variety

$$I(r,d,g) = \{(S,X) : X \text{ is smooth and } S \subset X\} \subset H(d,g) \times \mathbb{P}(r) \tag{3.2}$$

with the two obvious projections $p(r) : I(r,d,g) \to H(d,g)$ and $q(r) : I(r,d,g) \to \mathbb{P}(r)$. The fibers of $q(r)$ are projective spaces of fixed dimension (by Riemann-Roch).

We will show that $I(r,d,g)$ does not dominate $\mathbb{P}(r)$ for all choices of d,g corresponding to surfaces associated with an indecomposable ACM rank 2 bundle on a general X_r. This is achieved in the next sections by computing the dimension of $I(r,d,g)$ and observing that it is smaller than $\dim(\mathbb{P}(r))$.

Let us see, for instance, what happens for quadric surfaces.

Remark 3.7. Any quadric surface S contained in a general hypersurface X_r, $r \geq 3$, is reduced since X_r contains no planes. Hence it is a surface in $\mathbb{P}^3$, that is S is a complete intersection of type $(1,1,2)$ in $\mathbb{P}^5$.

Thus we may compute the normal bundle N_S of S:

$$h^0(N_S) = h^0(\mathcal{O}_S(2) \oplus \mathcal{O}_S(1) \oplus \mathcal{O}_S(1)) = 9+4+4 = 17$$

hence $\dim(H(2,0)) \leq 17$.

Proposition 3.8. *On a general hypersurface $X_r \subset \mathbb{P}^5$ of degree $r \geq 3$ there are no indecomposable normalized ACM rank 2 bundles $\mathcal{E}$ with $c_1(\mathcal{E}) = 4 - r$.*

Proof. First we show that any such bundle $\mathcal{E}$ is associated with a complete intersection quadric surface.
Consider a global section $s \in H^0(\mathcal{E})$ and its zero-locus S. The exact sequence (2.3) here reads

$$0 \to \mathcal{O}_{X_r} \to \mathcal{E} \to \mathcal{I}_{S/X_r}(4-r) \to 0$$

and implies $h^0(\mathcal{E}(r-4)) = h^0(\mathcal{O}_{X_r}(r-4))$. $h^1(\mathcal{E}(r-4))$ and $h^2(\mathcal{E}(r-4))$ vanish by assumptions. By duality $h^4(\mathcal{E}(r-4)) = h^0(\mathcal{E}(r-6)) = h^0(\mathcal{O}_{X_r}(r-6))$. Hence just as in proposition 3.3 we use Riemann-Roch to prove that $c_2(\mathcal{E}) = 2$. So S has degree 2. Since a general X_r contains no planes, S is reduced and the claim is proved.
Let $\mathcal{I}_S$ indicate the ideal sheaf of S in $\mathbb{P}^5$. One computes from the resolution of $\mathcal{I}_S$:

$$h^0(\mathcal{I}_S(r)) = 2h^0(\mathcal{O}_{\mathbb{P}^5}(r-1)) - 2h^0(\mathcal{O}_{\mathbb{P}^5}(r-3)) + h^0(\mathcal{O}_{\mathbb{P}^5}(r-4))$$

and thus one easily sees that:

$$\dim(I(r,2,0)) \leq h^0(\mathcal{I}_S(r)) - 1 + h^0(N_S) < \dim(\mathbb{P}(r))$$

for $r > 2$, which means that the map $q(r)$ above is not dominant. The conclusion follows. $\square$

With the results above we dispose of the case of cubic hypersurfaces:

Proposition 3.9. *On a general hypersurface $X := X_3 \subset \mathbb{P}^5$ of degree 3 there are no indecomposable ACM rank 2 bundles.*

Proof. By proposition 3.3 we know that any normalized indecomposable ACM rank 2 bundle on a smooth cubic hypersurface satisfies $3 > c_1(\mathcal{E}) > 0$. So only the cases $c_1(\mathcal{E}) = 1$ and $c_1(\mathcal{E}) = 2$ are left. But on a general cubic hypersurface the case $c_1(\mathcal{E}) = 2$ is excluded by proposition 3.5 while the case $c_1(\mathcal{E}) = 1$ is excluded by proposition 3.8. $\square$

4. Quartic hypersurfaces

In this section we fix $r = 4$. Our goal is to exclude the existence of indecomposable ACM rank 2 bundles $\mathcal{E}$ on a general quartic fourfold $X := X_4$. We also assume that $\mathcal{E}$ is normalized.

Arguing as in proposition 3.9 we know that for such a bundle $\mathcal{E}$ the only possibilities for the first Chern classes are $c_1(\mathcal{E}) = 2$ and $c_1(\mathcal{E}) = 1$.

We dispose of these cases using a computation for the normal bundle of the zero–locus of a general global section of $\mathcal{E}$.

Remark 4.1. If $\mathcal{E}$ is an ACM rank 2 bundle on a smooth hypersurface X_r, then the zero–locus S of a global section of $\mathcal{E}$ has codimension at most 3 in $\mathbb{P}^5$. If it has codimension 3, then it is an "arithmetically Gorenstein" subscheme of codimension 3 in the projective space $\mathbb{P}^5$. Thus its ideal sheaf $\mathcal{I}_S$ in $\mathbb{P}^5$ has a self dual free resolution of type

$$0 \to \mathcal{O}(-e-6) \to \bigoplus_{j=1}^{r} \mathcal{O}(-m_j) \to \bigoplus_{i=1}^{r} \mathcal{O}(-n_i) \to \mathcal{I}_S \to 0 \tag{4.1}$$

where e is the number such that the canonical class of S is e times the hyperplane section and $e + 6 - m_i = n_i$ for all i.

Using the previous resolution one can compute the cohomology of the normal bundle N_S of S in $\mathbb{P}^5$. Indeed by [9] and Theorem 2.6 of [11] we have the following:

Proposition 4.2. (Kleppe - Miró-Roig) *With the notation of the previous remark, order the integers n_i and m_j so that:*

$$n_1 \leq n_2 \leq ... \leq n_r \quad \textit{and} \quad m_1 \geq m_2 \geq ... \geq m_r.$$

Then:

$$
\begin{aligned}
h^0 N_S = {} & \sum_{i=1}^{r} h^0 \mathcal{O}_S(n_i) + \sum_{1\le i<j\le r} \binom{-n_i+m_j+5}{5} + \\
& - \sum_{1\le i<j\le r} \binom{n_i-m_j+5}{5} - \sum_{i=1}^{r} \binom{n_i+5}{5}.
\end{aligned}
\tag{4.2}
$$

Remark 4.3. If S is an ACM subscheme of $\mathbb{P}^5$ and C is a general hyperplane section of S, then a minimal resolution of the ideal sheaf of C in $\mathbb{P}^4$ lifts to a minimal resolution of the ideal sheaf of S in $\mathbb{P}^5$.

Let us go back to general quartic hypersurfaces X.

A general hyperplane section Y of X is a general quartic threefold in $\mathbb{P}^4$ and the restriction $\mathcal{E}'$ of $\mathcal{E}$ to Y is an indecomposable ACM bundle of rank 2. These bundles are classified in [13], where the possibilities for the second Chern classes of $\mathcal{E}'$, hence also of $\mathcal{E}$, are listed. These possibilities are:

$$(c_1, c_2) \in \{(-1,1),(0,2),(1,3),(1,4),(1,5),(2,8),(3,14)\}.$$

The cases $(c_1, c_2) = (-1,1),(0,2),(3,14)$ cannot occur on a general quartic fourfold, by propositions 3.5, 3.3 and 3.8.

We explore the remaining possibilities case by case.

Case 4.1. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 3$.
By [13] $\mathcal{E}'$ is associated with a plane cubic curve, hence $\mathcal{E}$ is associated with a cubic surface $S \subset \mathbb{P}^3$. It turns out $h^0(N_S) = h^0(\mathcal{O}_S(3) \oplus \mathcal{O}_S^2(1)) = 27$ while the ideal sheaf $\mathcal{I}_S$ has $h^0(\mathcal{I}_S(4)) = 95$. Thus in this case $\dim(I(4,3,1)) \le 121$. Since $\dim(\mathbb{P}(4)) = 125$, the projection $q(4) : I(4,3,1) \to \mathbb{P}(4)$ cannot be dominant and this case is excluded on a general quartic hypersurface X.

Case 4.2. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 4$.
By [13] $\mathcal{E}'$ is associated with a quartic curve, complete intersection of 2 quadrics in $\mathbb{P}^3$. Hence $\mathcal{E}$ is associated with a complete intersection of two quadrics $S \subset \mathbb{P}^4$. It turns out $h^0(N_S) = h^0(\mathcal{O}_S^2(2) \oplus \mathcal{O}_S(1)) = 31$ while $h^0(\mathcal{I}_S(4)) = 85$, so that $\dim(I(4,4,1)) \le 115$ and the projection $q(4) : I(4,4,1) \to \mathbb{P}(4)$ cannot be dominant.

Case 4.3. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 5$.
By [13] $\mathcal{E}'$ is associated to an elliptic quintic curve, whose ideal sheaf $\mathcal{I}$ in $\mathbb{P}^4$ has resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-5) \to \mathcal{O}_{\mathbb{P}^4}^5(-3) \to \mathcal{O}_{\mathbb{P}^4}^5(-2) \to \mathcal{I} \to 0 \tag{4.3}$$

from which we have the resolution for the ideal sheaf $\mathcal{I}_S$ of a quintic surface S associated with $\mathcal{E}$. Now we use proposition 4.2 to compute $h^0(N_S) = 35$ while from the resolution we get $h^0(\mathcal{I}_S(4)) = 75$ so that $\dim(I(4,5,1)) \le 109$ and again $q(4)$ does not dominate $\mathbb{P}(4)$.

Case 4.4. $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 8$.
By [13] $\mathcal{E}'$ is associated to a curve of degree 8 in $\mathbb{P}^4$, whose ideal sheaf $\mathcal{I}$ has resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-6) \to \mathcal{O}_{\mathbb{P}^4}^3(-4) \oplus \mathcal{O}_{\mathbb{P}^4}^x(-3) \to \mathcal{O}_{\mathbb{P}^4}^x(-3) \oplus \mathcal{O}_{\mathbb{P}^4}^3(-2) \to \mathcal{I} \to 0 \tag{4.4}$$

from which we have the resolution for the ideal sheaf $\mathcal{I}_S$ of a surface S of degree 8 associated with $\mathcal{E}$. Notice that we do not know the number of minimal generators

of degree 3 for the ideal sheaf of S (if any). Nevertheless we may use proposition 4.2 to compute $h^0(N_S)$. Indeed in the computation it turns out that the contribution of cubic generators disappears and one gets $h^0(N_S) = 54$. Also one sees that $h^0(\mathcal{I}_S(4)) = 60$ so that $\dim(I(4,8,5)) \leq 113$ and again $q(4)$ does not dominate $\mathbb{P}(4)$.

No other cases may occur, by [13]. Hence we conclude:

Proposition 4.4. *On a general hypersurface $X \subset \mathbb{P}^5$ of degree 4 there are no indecomposable ACM rank 2 bundles.*

5. Quintic hypersurfaces

In this section we exclude the existence of indecomposable ACM rank 2 bundles $\mathcal{E}$ on a general quintic fourfold X. As usual we assume that $\mathcal{E}$ is normalized.

In this case, we are left with several cases for the first Chern class, namely $c_1(\mathcal{E}) = 0, 1, 2, 3$.

Again a general hyperplane section Y of X is a general quintic threefold in $\mathbb{P}^4$ and the restriction $\mathcal{E}'$ of $\mathcal{E}$ to Y is an indecomposable ACM bundle of rank 2. These bundles are classified in [5]. In particular for the Chern classes we have the following possibilities:

c_1	c_2
0	$3, 4, 5$
1	$4, 6, 8$
2	$11, 12, 13, 14$
3	20

We explore again the situation case by case.

Case 5.1. $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 3$.
By [5] $\mathcal{E}'$ is associated to a plane cubic curve, hence $\mathcal{E}$ is associated to a cubic surface $S \subset \mathbb{P}^3$. Then as above one computes $h^0(N_S) = 27$ while the ideal sheaf $\mathcal{I}_S$ has $h^0(\mathcal{I}_S(5)) = 206$. Thus in this case $\dim(I(5,3,1)) \leq 232$. Since $\dim(\mathbb{P}(5)) = 251$, the projection $q(5) : I(5,3,1) \to \mathbb{P}(5)$ cannot be dominant and this case is excluded on a general quintic hypersurface.

Case 5.2. $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 4$.
By [5] $\mathcal{E}'$ is associated with a quartic curve, complete intersection of 2 quadrics in $\mathbb{P}^3$. Hence $\mathcal{E}$ is associated to a complete intersection of two quadrics $S \subset \mathbb{P}^4$. It turns out $h^0(N_S) = 31$ while $h^0(\mathcal{I}_S(5)) = 191$, so that $\dim(I(5,4,1)) \leq 221$ and $q(5)$ is not dominant.

Case 5.3. $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = 5$.
By [5] $\mathcal{E}'$ is associated with a quintic elliptic curve and as above one gets a resolution

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-5) \to \mathcal{O}^5_{\mathbb{P}^5}(-3) \to \mathcal{O}^5_{\mathbb{P}^5}(-2) \to \mathcal{I} \to 0 \tag{5.1}$$

for the ideal sheaf of a surface associated with $\mathcal{E}$. Then $h^0(N_S) = 35$ while $h^0(\mathcal{I}_S(5)) = 176$, so that $\dim(I(5,5,1)) \leq 210$ and $q(5)$ is not dominant.

Case 5.4. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 4$.
By [5] $\mathcal{E}'$ is associated with a plane quartic curve, hence $\mathcal{E}$ is associated with a quartic surface $S \subset \mathbb{P}^3$. It turns out $h^0(N_S) = h^0(\mathcal{O}_S(4) \oplus \mathcal{O}^2_S(1)) = 42$ while the

ideal sheaf $\mathcal{I}_S$ has $h^0(\mathcal{I}_S(5)) = 200$. Thus in this case $\dim(I(5,4,3)) \leq 241$ and the projection $q(5) : I(5,4,3) \to \mathbb{P}(5)$ cannot be dominant.

Case 5.5. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 6$.
By [5] $\mathcal{E}'$ is associated with a sextic curve, complete intersection of a quadric and a cubic in $\mathbb{P}^3$. Hence $\mathcal{E}$ is associated with a corresponding complete intersection $S \subset \mathbb{P}^4$. It turns out $h^0(N_S) = 48$ while $h^0(\mathcal{I}_S(5)) = 175$, so that $\dim(I(5,6,4)) \leq 222$ and the projection $q(5)$ cannot be dominant.

Case 5.6. $c_1(\mathcal{E}) = 1$, $c_2(\mathcal{E}) = 8$.
By [5] $\mathcal{E}'$ is associated to a curve of degree 8 in $\mathbb{P}^4$, whose ideal sheaf $\mathcal{I}$ has resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-6) \to \mathcal{O}^3_{\mathbb{P}^4}(-4) \oplus \mathcal{O}^x_{\mathbb{P}^4}(-3) \to \mathcal{O}^x_{\mathbb{P}^4}(-3) \oplus \mathcal{O}^3_{\mathbb{P}^4}(-2) \to \mathcal{I} \to 0. \tag{5.2}$$

As above one gets $h^0(N_S) = 54$ while $h^0(\mathcal{I}_S(5)) = 150$ so that $\dim(I(5,8,5)) \leq 203$ and $q(5)$ is not dominant.

Consider now the case $c_1(\mathcal{E}) = 2$ and $c_2(\mathcal{E}) = 11, 12, 13, 14$. Let S be a surface associated with $\mathcal{E}$ and call C a general hyperplane section of S, which is thus associated with $\mathcal{E}'$. One computes:

$$h^0(\mathcal{I}_S(5)) = 245 - 10\deg(S) = 245 - 10c_2(\mathcal{E})$$

so that we only need to prove that:

$$h^0(N_S) < 10c_2(\mathcal{E}) + 7. \tag{5.3}$$

We use the results of [5] §4 and [6] case 5.7 to compute a minimal resolution for the ideal sheaf of C in $\mathbb{P}^4$, hence also a resolution of $\mathcal{I}_S$, which leads to the computation of $h^0(N_S)$, via proposition 4.2.

Case 5.7. $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 11$.
By [5] §4 the resolution of the ideal sheaf $\mathcal{I}_S$ is:

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \mathcal{O}^b_{\mathbb{P}^5}(-3) \oplus \mathcal{O}^c_{\mathbb{P}^5}(-4) \oplus \mathcal{O}^3_{\mathbb{P}^5}(-5) \to \\ \to \mathcal{O}^3_{\mathbb{P}^5}(-2) \oplus \mathcal{O}^c_{\mathbb{P}^5}(-3) \oplus \mathcal{O}^b_{\mathbb{P}^5}(-4) \to \mathcal{I}_S \to 0. \end{aligned} \tag{5.4}$$

Comparing the first Chen classes in the exact sequence, one finds $c = b - 2$. Using equation 4.2 one is able to compute $h^0(N_S)$. It turns out that b and c cancel and one finds $h^0(N_S) = 83 < 117$ so that $\dim(I(5,11,12)) \leq 214$ and $q(5)$ is not dominant.

Case 5.8. $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 12$.
By [5] §4 the resolution of the ideal sheaf $\mathcal{I}_S$ is:

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \mathcal{O}^b_{\mathbb{P}^5}(-3) \oplus \mathcal{O}^c_{\mathbb{P}^5}(-4) \oplus \mathcal{O}^2_{\mathbb{P}^5}(-5) \to \\ \to \mathcal{O}^2_{\mathbb{P}^5}(-2) \oplus \mathcal{O}^c_{\mathbb{P}^5}(-3) \oplus \mathcal{O}^b_{\mathbb{P}^5}(-4) \to \mathcal{I}_S \to 0 \end{aligned} \tag{5.5}$$

where $b = c - 1$ and $b = 0, 1$, according with the existence of a cubic syzygy between the two quadrics. In both cases, using equation 4.2 one computes $h^0(N_S) = 81 < 127$ so that $\dim(I(5,12,13)) \leq 205$ and $q(5)$ is not dominant.

Case 5.9. $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 13$.
In this case we have only one quadric containing S and the resolution of $\mathcal{I}_S$ is given by:

$$\begin{aligned} 0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \oplus \mathcal{O}^4_{\mathbb{P}^5}(-4) \oplus \mathcal{O}_{\mathbb{P}^5}(-5) \to \\ \to \mathcal{O}_{\mathbb{P}^5}(-2) \oplus \mathcal{O}^4_{\mathbb{P}^5}(-3) \to \mathcal{I}_S \to 0. \end{aligned} \tag{5.6}$$

So one computes $h^0(N_S) = 79 < 137$. It follows that $q(5)$ is not dominant.

Case 5.10. $c_1(\mathcal{E}) = 2$, $c_2(\mathcal{E}) = 14$.
By [5] §4 the resolution of the ideal sheaf $\mathcal{I}_S$ is:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-7) \to \oplus\mathcal{O}^7_{\mathbb{P}^5}(-4) \to \oplus\mathcal{O}^7_{\mathbb{P}^5}(-3) \to \mathcal{I}_S \to 0 \tag{5.7}$$

and one computes $h^0(N_S) = 77 < 147$ so that $q(5)$ is not dominant.

Finally for $c_1 = 3$ we have:

Case 5.11. $c_1(\mathcal{E}) = 3$, $c_2(\mathcal{E}) = 20$.
By [5] we know the resolution of the ideal sheaf of a curve associated with $\mathcal{E}'$, so that the ideal sheaf of a surface associated with $\mathcal{E}$ is:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-8) \to \mathcal{O}^4_{\mathbb{P}^5}(-5) \to \mathcal{O}^4_{\mathbb{P}^5}(-3) \to \mathcal{I}_S \to 0. \tag{5.8}$$

One computes $h^0(N_S) = 110$ and $h^0(\mathcal{I}_S(5)) = 80$ so that $\dim(I(5, 20, 31)) \leq 189$ and $q(5)$ is not dominant.

Hence we may conclude

Proposition 5.1. *On a general hypersurface $X \subset \mathbb{P}^5$ of degree 5 there are no indecomposable ACM rank 2 bundles.*

The main theorem follows.

Remark 5.2. By [4] there exists a non discrete family (up to twist) of isomorphism classes of indecomposable ACM vector bundles on any smooth projective hypersurface of degree $r \geq 3$ in the 5–dimensional complex projective space $\mathbb{P}^5$. On a general X_r the rank of the bundles constructed in [4] is 16 (cfr. [14]).

The problem of determining the minimum rank $BGS(X_r)$ for ACM bundles on X_r moving in a non–trivial family (the *BGS invariant)* is still open.

We prove in this paper that $BGS(X_r) > 2$ for general hypersurfaces in $\mathbb{P}^5$ of degree $r \leq 6$.

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